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Electromagnetic radiation recoil

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Abstract. Electromagnetic waves from a finite oscillating linear cohesive distribution of electric charge are investigated by a double-parameter approximation method applied to the Einstein–Maxwell equations of general relativity. In the metric representing the distribution, gravitational dipole terms (proportional to time) appear which indicate that the source recoils owing to the linear momentum carried away from the source by the waves.

1. Introduction

It was shown by Rotenberg (1966) that an oscillating electric dipole emitting electromagnetic waves suffers a permanent loss of mass, on account of the energy of radiation transmitted from the source as calculated by means of the electromagnetic energy tensor. In a later work (Rotenberg 1975), this result was extended for any isolated cohesive source of electromagnetic waves; in that article it was also shown that the source, if rotating, undergoes steady diminution of angular momentum. The object of the present paper is to establish that a finite oscillating linear cohesive distribution of electric charge generally recoils away to infinity (as does a similar distribution of mass: see Bonnor and Rotenberg 1966), and that this is due to the linear momentum of radiation transmitted from the source as calculated via the electromagnetic energy tensor. To achieve this we shall use a double-parameter approximation method presented in § 4, similar in form to that invented by Bonnor (Bonnor 1959, Bonnor and Rotenberg 1966) and appearing in the above mentioned work (Rotenberg 1966). As in the latter paper, the approximation method will be applied to the metric tensor and to the Einstein–Maxwell equations†

$$
\begin{aligned}
&R_{ik} = -8\pi E_{ik} \qquad && E^i_k = -F^{ia}F_{ka} + \tfrac{1}{4}\delta^i_k F^{ab}F_{ab} \\
&F^{ia}{}_{;a} = 0 \qquad && F_{ik} = \phi_{i,k} - \phi_{k,i}
\end{aligned}
\tag{1.1}
$$

for free space (Eddington 1924, §§ 73 and 77), in which ϕ_i, F_{ik} and E_{ik} are the electromagnetic 4-potential, 4×4-field and energy tensors, respectively. To reduce calculations, coordinates of the Bondi metric (see § 5) will be utilized to carry out this method.

† In this paper, Latin indices run from 1 to 4 and Greek indices run from 1 to 3; the summation convention applies to both types of indices. Comma subscripts denote partial differentiation and semicolon subscripts indicate covariant differentiation.

The plan of the paper is as follows. After a further brief description of the linear source in § 2, the external multipole wave solution for ϕ_i of the linearized Einstein–Maxwell equations is obtained in § 3 for outgoing electromagnetic waves from the linear source; this solution corresponds to (pseudo-) Galilean coordinates, to be referred to as $x_i = (x, y, z, t) = (x_\alpha, t)$, with origin O. In § 4, the double-parameter approximation method is introduced, and the metric invented by Bondi (1960) is presented in § 5, where, in addition, the external multipole wave solution in the coordinates of this metric is deduced from the one of § 3 in Galilean coordinates. The multipole wave solution in the Bondi metric is needed in § 6 to calculate the electromagnetic energy tensor in this metric and the total linear momentum transported by the waves from their linear source. Finally, in § 7, gravitational dipole terms (proportional to the time t) are found in the solution of the approximate Einstein–Maxwell equations and are shown to lead to the following result. The source generally undergoes a secular change in linear momentum which is equal and opposite to the linear momentum carried away from the source as radiation; in this case the source recoils off to infinity. The more complicated calculations are relegated to the appendices.

2. The electromagnetic source

For choosing a source of electromagnetic waves we shall consider a linear coherent distribution of electric charge of finite length along the axis Oz of a (pseudo-) rectangular Cartesian coordinate system, in which the origin O coincides with the centre of mass of the distribution†. The source oscillates arbitrarily but smoothly during a finite period $t_1 \leqslant t \leqslant t_2$ so that $\underset{s}{I}(t)$, the sth moment of charge at time t of the source about O, is an arbitrary bounded function with unique derivatives of all orders in the interval $t_1 \leqslant t \leqslant t_2$ and is constant outside this interval.

In terms of the retarded time $u = t - r$ used later as the time-like coordinate for the Bondi metric, the period of motion will be referred to as $u_1 \leqslant u \leqslant u_2$.

3. The outgoing multipole wave solution of the linearized Einstein–Maxwell equations

For ϕ_i representing any outgoing electromagnetic wave field, we present below, after introducing convenient notation, the external multipole wave solution of the linearized form of the second pair of equations (1.1) or of

$$F^{ia}{}_{;a} = 4\pi J^i \qquad F_{ik} = \phi_{i,k} - \phi_{k,i} \tag{3.1}$$

J_i being the 4-current density of the source of the field; the solution, expressed in Galilean coordinates $x_i = (x_\alpha, t)$, is derived in appendix 1. We then apply this solution to the special system of § 2.

Let e be the total charge of the source of the field, so that

$$e = \int_V J_4 \, dx_1 \, dx_2 \, dx_3 \tag{3.2}$$

† It will be assumed that distance, time and mass retain their Newtonian significance in the linear approximation to the Einstein–Maxwell equations.

where V is any space volume including the source; and let a be a constant having the dimension of length and characterizing the extent of the source by representing, for example, the time-averaged radius of gyration of the source. Let

$$I_{i:\sigma\rho\tau\ldots}(t) \overset{\text{def}}{=} \int_V x_\sigma x_\rho x_\tau \ldots J_i(x_\alpha, t)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 \tag{3.3}$$

represent the moments at time t of the 4-current density J_i for the source about the coordinate planes $x_\alpha = 0$; these moments must therefore satisfy the conservation law

$$\eta^{ab} J_{a,b} = 0 \qquad \eta^{ik} = \eta_{ik} \overset{\text{def}}{=} \mathrm{diag}(-1, -1, -1, +1) \tag{3.4}$$

for J_i. Then introduce the *specific* moments, unaffected by change of units for e or a, as

$$\begin{aligned} h_{\alpha:\sigma_1\sigma_2\ldots\sigma_s} &\overset{\text{def}}{=} e^{-1} a^{-s-1} I_{\alpha:\sigma_1\sigma_2\ldots\sigma_s} \\ h_{4:\sigma_1\sigma_2\ldots\sigma_s} &\overset{\text{def}}{=} e^{-1} a^{-s} I_{4:\sigma_1\sigma_2\ldots\sigma_s}. \end{aligned} \tag{3.5}$$

Finally, let $r \overset{\text{def}}{=} (x_\sigma x_\sigma)^{1/2}$ be the (pseudo-) radius vector OP of the field point P with (pseudo-) spherical polar coordinates (r, θ, ϕ) and write

$$n_\alpha \overset{\text{def}}{=} x_\alpha / r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{3.6}$$

In the above notation, the exterior multipole wave solution (for outgoing waves) of the linear approximation to equations (3.1) is (appendix 1)

$$\begin{aligned} \phi_\alpha &= e\{ a r^{-1} h_\alpha + a^2 n_\sigma (r^{-1} h'_{\alpha:\sigma} + r^{-2} h_{\alpha:\sigma}) \\ &\quad + a^3[\tfrac{1}{2} r^{-1} n_\sigma n_\rho h''_{\alpha:\sigma\rho} + \tfrac{1}{2}(3 n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2} h'_{\alpha:\sigma\rho} + r^{-3} h_{\alpha:\sigma\rho})] + \mathrm{O}(a^4)\} \\ \phi_4 &= e\{ r^{-1} + a n_\sigma (r^{-1} h'_{4:\sigma} + r^{-2} h_{4:\sigma}) \\ &\quad + a^2[\tfrac{1}{2} r^{-1} n_\sigma n_\rho h''_{4:\sigma\rho} + \tfrac{1}{2}(3 n_\sigma n_\rho - \delta_{\sigma\rho})(r^{-2} h'_{4:\sigma\rho} + r^{-3} h_{4:\sigma\rho})] \\ &\quad + a^3[\tfrac{1}{6} r^{-1} n_\sigma n_\rho n_\tau h'''_{4:\sigma\rho\tau} + \tfrac{1}{2} r^{-2} n_\sigma (2 n_\rho n_\tau - \delta_{\rho\tau}) h''_{4:\sigma\rho\tau} \\ &\quad + \tfrac{1}{2} n_\sigma (5 n_\rho n_\tau - 3\delta_{\rho\tau})(r^{-3} h'_{4:\sigma\rho\tau} + r^{-4} h_{4:\sigma\rho\tau})] + \mathrm{O}(a^4)\}; \end{aligned} \tag{3.7}$$

here $h_{i:\sigma\rho\tau\ldots}$ are to be evaluated at (pseudo-) retarded time $u \overset{\text{def}}{=} t - r$ and a prime denotes differentiation with respect to the argument u.

In the solution (3.7) the 2^s*-pole wave* ($s = 0, 1, 2, \ldots$) is the part involving ea^s; only the monopole contribution and the dipole, quadrupole and octupole wave contributions have been explicitly shown in this multipole wave solution, as these are sufficient to obtain the results of § 7.

To deduce from equations (3.7) the particular form of the multipole wave solution for the linear emitter in § 2, we proceed in the following way. For this source, we now consider the 4-current density J_i as a linear density $J_i(z, t)$ with

$$J_1 = J_2 = 0. \tag{3.8}$$

In the notations (3.3) and (3.5) we now have

$$h_{i:\sigma\rho\tau\ldots}=0 \quad \text{when } i \text{ or } \sigma \text{ or } \rho \text{ or } \tau \text{ or } \ldots = 1 \text{ or } 2 \tag{3.9}$$

$$\begin{aligned} h_3 &= e^{-1}a^{-1}\int_{-\infty}^{\infty} J_3(z,t)\,\mathrm{d}z \\ h_{3:3} &= e^{-1}a^{-2}\int_{-\infty}^{\infty} zJ_3(z,t)\,\mathrm{d}z \\ h_{3:33} &= e^{-1}a^{-3}\int_{-\infty}^{\infty} z^2J_3(z,t)\,\mathrm{d}z \end{aligned} \tag{3.10}$$

and

$$h_{4:3}=\overset{1}{k} \qquad h_{4:33}=\overset{2}{k} \qquad h_{4:333}=\overset{3}{k} \tag{3.11}$$

in which

$$\overset{s}{k} \overset{\text{def}}{=} e^{-1}a^{-s}\overset{s}{I} \qquad \overset{s}{I}(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} z^s J_4(z,t)\,\mathrm{d}z \qquad (s=0,1,2,\ldots) \tag{3.12}$$

$\overset{s}{I}$ being, as in § 2, the sth moment of the charge of the linear source about the origin O. The conservation equation (3.4) now reads

$$J_{3,3}=J_{4,4} \tag{3.13}$$

owing to equations (3.8). Multiplying this by z^{s+1} and integrating along Oz between the limits $z=\pm\infty$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} z^{s+1}J_4\,\mathrm{d}z &= \int_{-\infty}^{\infty} z^{s+1}J_{4,4}\,\mathrm{d}z = \int_{-\infty}^{\infty} z^{s+1}J_{3,3}\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \{(z^{s+1}J_3)_{,3}-(s+1)z^sJ_3\}\,\mathrm{d}z \\ &= [z^{s+1}J_3]_{-\infty}^{\infty}-(s+1)\int_{-\infty}^{\infty} z^sJ_3\,\mathrm{d}z = -(s+1)\int_{-\infty}^{\infty} z^sJ_3\,\mathrm{d}z \end{aligned}$$

so that

$$\int_{-\infty}^{\infty} z^sJ_3\,\mathrm{d}z = -\frac{1}{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} z^{s+1}J_4\,\mathrm{d}z. \tag{3.14}$$

In the notations (3.10) and (3.12) this yields

$$h_3=-\overset{1}{k}' \qquad h_{3:3}=-\tfrac{1}{2}\overset{2}{k}' \qquad h_{3:33}=-\tfrac{1}{3}\overset{3}{k}' \tag{3.15}$$

where a prime indicates differentiation with respect to the argument t.

Using equations (3.8) in the first of equations (A.1) of appendix 1 and using equations (3.6), (3.9), (3.11) and (3.15) in equations (3.7) lead to the following exterior multipole wave solution for the linear emitter in § 2:

$$\phi_1 = \phi_2 = 0$$

$$\begin{aligned}\phi_3 = e\{&-ar^{-1}\overset{1}{k}{}' - \tfrac{1}{2}a^2c(r^{-1}\overset{2}{k}{}'' + r^{-2}\overset{2}{k}{}') \\ &+a^3[-\tfrac{1}{6}r^{-1}c^2\overset{3}{k}{}''' - \tfrac{1}{6}(2-3s^2)(r^{-2}\overset{3}{k}{}'' + r^{-3}\overset{3}{k}{}')] + \mathrm{O}(a^4)\}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\phi_4 = e\{&r^{-1} + ac(r^{-1}\overset{1}{k}{}' + r^{-2}\overset{1}{k}) + a^2[\tfrac{1}{2}r^{-1}c^2\overset{2}{k}{}'' + \tfrac{1}{2}(2-3s^2)(r^{-2}\overset{2}{k}{}' + r^{-3}\overset{2}{k})] \\ &+a^3[\tfrac{1}{6}r^{-1}c^3\overset{3}{k}{}''' + \tfrac{1}{2}r^{-2}(-c+2c^3)\overset{3}{k}{}'' + \tfrac{1}{2}(-3c+5c^3)(r^{-3}\overset{3}{k}{}' + r^{-4}\overset{3}{k})] + \mathrm{O}(a^4)\}.\end{aligned}$$

In this,

$$s \overset{\text{def}}{=} \sin\theta \qquad c \overset{\text{def}}{=} \cos\theta; \tag{3.17}$$

$\overset{s}{k}$ are to be calculated at retarded time u, and a prime means differentiation with respect to u.

4. The double-parameter approximation method

We shall suppose that the metric tensor g_{ik} and the electromagnetic energy tensor E_{ik} representing the external electromagnetic field can be expanded as convergent double-series in ascending powers of the parameters e and a introduced in § 3, in the following manner:

$$g_{ik} = \overset{(00)}{g_{ik}} + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{g_{ik}} \tag{4.1}$$

$$E_{ik} = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{E_{ik}} \tag{4.2}$$

with $\overset{(ps)}{g_{ik}}$ and $\overset{(ps)}{E_{ik}}$ involving the coordinates x^i of the system employed and $\overset{(00)}{g_{ik}}$ referring to flat space–time (see Rotenberg 1975). The contravariant metric tensor g^{ik} will then have a similar expansion, namely

$$g^{ik} = \overset{(00)}{g^{ik}} + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{g^{ik}} \tag{4.3}$$

with $\overset{(ps)}{g^{ik}}$ involving the coordinates x^i of the system employed and $\overset{(00)}{g^{ik}}$ referring to flat space–time.

The reason for commencing the summation with respect to p in each of the equations (4.1)–(4.3) from $p=2$, rather than from $p=1$, is as follows. Using the expansions (4.1) and (4.3) for g_{ik} and g^{ik} along with equations (3.7) and the fourth of equations (1.1) in the second of equations (1.1) will give rise to the expansion (4.2) for E_{ik}. In this expansion, the summation with respect to p starts with $p=2$, and it can easily be seen that this would remain the case even if the range of summation in the expansions (4.1) and (4.3) for g_{ik} and g^{ik} were extended to include $p=1$. So we may as well allow the summation with respect to p for g_{ik} and g^{ik} to begin with $p=2$ as in the expansion for E_{ik}, since the metric depends, to some extent, on the electromagnetic field represented by E_{ik}.

The double-series expansions (4.1) and (4.2) will now constitute the double-parameter approximation method for investigating the effect of electromagnetic waves on the linear momentum of their source. Inserting these expansions in the first of equations (1.1) and comparing the coefficients of $e^p a^s$ on both sides for each given pair $p, s = 0, 1, 2, \ldots$, we obtain ten second-order differential equations of the form

$$\Phi_{lm}(\overset{(ps)}{g}_{ik}) = \overset{(ps)}{\Psi}_{lm}(\overset{(qr)}{g}_{ik}) + \text{constant} \times \overset{(ps)}{E}_{lm} \qquad (2 \leqslant q \leqslant p-1 \qquad 0 \leqslant r \leqslant s) \tag{4.4}$$

henceforth referred to as the (ps) *approximation*. The left-hand sides, Φ_{lm}, are linear in $\overset{(ps)}{g}_{ik}$ and their derivatives; the quantities $\overset{(ps)}{\Psi}_{lm}$ on the right-hand sides are non-linear in $\overset{(qr)}{g}_{ik}$ and their derivatives, known from earlier approximations. Thus, apart from the expressions involving $\overset{(2s)}{E}_{lm}$, the $(2s)$ approximations contain only terms linear in $\overset{(2s)}{g}_{ik}$ and their derivatives; the non-linear $\overset{(2s)}{\Psi}_{lm}$ are absent from the $(2s)$ approximations. These $(2s)$ approximations $(s = 0, 1, 2, 3)$ are the only ones considered in this paper and calculated for the linear source in § 2. Indeed, it is in the (23) approximation that there first appear gravitational dipole terms† which reveal that the linear source undergoes a generally permanent change in linear momentum equal and opposite to the linear momentum removed from the source as radiation. Our object, therefore, is to derive appropriate solutions of the $(2s)$ approximations $(s = 0, 1, 2, 3)$.

Finally, the solution of the (ps) approximation will be simply referred to as the (ps) *solution*, represented by the $\overset{(ps)}{g}_{ik}$ satisfying equations (4.4).

5. The Bondi metric

To avoid excessive calculation in solving the leading $(2s)$ approximations (especially the (22) and (23) ones) for the special axi-symmetric source in § 2, we shall use the axi-symmetric metric of Bondi (1960), exhibited here in the form

$$ds^2 = -r^2(B\, d\theta^2 + C \sin^2\theta\, d\phi^2) + D\, du^2 + 2F\, dr\, du + 2Gr\, d\theta\, du \qquad C = B^{-1} \tag{5.1}$$

(as in Bonnor and Rotenberg 1966, Rotenberg 1966, Hunter and Rotenberg 1969, Rotenberg 1971); in this, B, C, D, F and G are functions of r, θ and u.

In the coordinates of the Bondi metric, flat space–time is represented by

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + du^2 + 2\, dr\, du \tag{5.2}$$

and the exterior Nordström solution assumes the form

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1 - 2mr^{-1} + 4\pi e^2 r^{-2})\, du^2 + 2\, dr\, du \tag{5.3}$$

(Rotenberg 1971), m being the mass of the central spherically symmetric body with centre the origin.

† For $u > u_2$, these terms are linear in u and take the form $r^{-2}cu$ or $r^{-2}su$ with numerical coefficients: see appendix 3.

The coefficients of the Bondi metric (5.1) may be expanded in forms similar to the expansion (4.1); thus

$$
\begin{aligned}
-r^{-2}g_{22} &= B = 1 + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{B} \\
-r^{-2}\operatorname{cosec}^2\theta\, g_{33} &= C = 1 + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{C} \\
g_{44} &= D = 1 + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{D} \\
g_{14} &= F = 1 + \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{F} \\
r^{-1}g_{24} &= G = \sum_{p=2}^{\infty}\sum_{s=0}^{\infty} e^p a^s \overset{(ps)}{G}.
\end{aligned}
\tag{5.4}
$$

Here, $\overset{(ps)}{B}$, $\overset{(ps)}{C}$, $\overset{(ps)}{D}$, $\overset{(ps)}{F}$ and $\overset{(ps)}{G}$ are functions of (r, θ, u), and $\overset{(ps)}{C}$ (with p, s given) is related to $\overset{(qr)}{B}$ $(2 \leqslant q \leqslant p,\ 0 \leqslant r \leqslant s)$ by the second of equations (5.1). The isolated terms 1 on the extreme right of equations (5.4) form the flat space–time metric (5.2), in which the following are the non-zero components $\overset{(00)}{g}_{ik}$:

$$
\overset{(00)}{g}_{22} = -r^2 \qquad \overset{(00)}{g}_{33} = -r^2\sin^2\theta \qquad \overset{(00)}{g}_{44} = 1 \qquad \overset{(00)}{g}_{14} = 1. \tag{5.5}
$$

The corresponding non-zero components $\overset{(00)}{g}{}^{ik}$ are

$$
\overset{(00)}{g}{}^{11} = -1 \qquad \overset{(00)}{g}{}^{22} = -r^{-2} \qquad \overset{(00)}{g}{}^{33} = -r^{-2}\operatorname{cosec}^2\theta \qquad \overset{(00)}{g}{}^{14} = 1. \tag{5.6}
$$

The notation (5.4) will be employed in § 7.

For ϕ_i corresponding to the linear source in § 2, we now obtain the exterior multipole wave solution (of the linear approximation to the second pair of equations (1.1)) in coordinates of the Bondi metric. To do this we apply the coordinate transformation

$$
x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \qquad t = u + r \tag{5.7}
$$

to equations (3.16); the result is

$$
\begin{aligned}
\phi_1 &= e\{r^{-1} + ar^{-2}c\overset{1}{k} + a^2[r^{-2}(\tfrac{1}{2} - s^2)\overset{2}{k}{}' + r^{-3}(1 - \tfrac{3}{2}s^2)\overset{2}{k}] \\
&\qquad + a^3[r^{-2}(-\tfrac{1}{3}c + \tfrac{1}{2}c^3)\overset{3}{k}{}'' + r^{-3}(-\tfrac{4}{3}c + 2c^3)\overset{3}{k}{}' + r^{-4}(-\tfrac{3}{2}c + \tfrac{5}{2}c^3)\overset{3}{k}] + \mathrm{O}(a^4)\} \\
\phi_2 &= e\{as\overset{1}{k}{}' + \tfrac{1}{2}a^2sc(\overset{2}{k}{}'' + r^{-1}\overset{2}{k}{}') \\
&\qquad + a^3[\tfrac{1}{6}sc^2\overset{3}{k}{}''' + (\tfrac{1}{3}s - \tfrac{1}{2}s^3)(r^{-1}\overset{3}{k}{}'' + r^{-2}\overset{3}{k}{}')] + \mathrm{O}(a^4)\} \\
\phi_3 &= 0 \\
\phi_4 &= e\{r^{-1} + ac(r^{-1}\overset{1}{k}{}' + r^{-2}\overset{1}{k}) \\
&\qquad + a^2[\tfrac{1}{2}r^{-1}c^2\overset{2}{k}{}'' + (1 - \tfrac{3}{2}s^2)(r^{-2}\overset{2}{k}{}' + r^{-3}\overset{2}{k})] \\
&\qquad + a^3[\tfrac{1}{6}r^{-1}c^3\overset{3}{k}{}''' + r^{-2}(-\tfrac{1}{2}c + c^3)\overset{3}{k}{}'' + (-\tfrac{3}{2}c + \tfrac{5}{2}c^3)(r^{-3}\overset{3}{k}{}' + r^{-4}\overset{3}{k})] + \mathrm{O}(a^4)\}
\end{aligned}
\tag{5.8}
$$

a prime attached to $\overset{s}{k}$ denoting differentiation with respect to the argument u. This solution will be required in the next section.

6. The electromagnetic energy tensor and the flux of linear momentum

To determine, for the linear source in § 2, the components E_{ik} of the electromagnetic energy tensor in the Bondi metric, we insert equations (5.8) in the fourth of equations (1.1), use the result and equations (4.3) and (5.6) in the second of equations (1.1), and then use equations (4.1) and (5.5). After lengthy but straightforward calculation we find

$$E_{ik}=e^2[\overset{(20)}{E}_{ik}+a\overset{(21)}{E}_{ik}+a^2\overset{(22)}{E}_{ik}+a^3\overset{(23)}{E}_{ik}+\mathrm{O}(a^4)]+\mathrm{O}(e^3) \tag{6.1}$$

where the non-zero $\overset{(20)}{E}_{ik}$ and $\overset{(21)}{E}_{ik}$ are given by

$$r^{-2}\overset{(20)}{E}_{22}=r^{-2}s^{-2}\overset{(20)}{E}_{33}=\overset{(20)}{E}_{44}=\overset{(20)}{E}_{14}=\tfrac{1}{2}r^{-4} \tag{6.2}$$

$$\begin{gathered} r^{-2}\overset{(21)}{E}_{22}=r^{-2}s^{-2}\overset{(21)}{E}_{33}=\overset{(21)}{E}_{44}=\overset{(21)}{E}_{14}=2c(r^{-4}\overset{1}{k}'+r^{-5}\overset{1}{k}) \\ r^{-1}\overset{(21)}{E}_{12}=-r^{-5}s\overset{1}{k} \qquad r^{-1}\overset{(21)}{E}_{24}=s(r^{-3}\overset{1}{k}''+r^{-4}\overset{1}{k}') \end{gathered} \tag{6.3}$$

where the non-zero $(10)\times(12)$ contributions to $\overset{(22)}{E}_{ik}$ are given by†

$$\begin{gathered} r^{-2}\overset{(22)}{E}_{22}=r^{-2}s^{-2}\overset{(22)}{E}_{33}=\overset{(22)}{E}_{44}=\overset{(22)}{E}_{14}=(1-\tfrac{3}{2}s^2)(r^{-4}\overset{2}{k}''+3r^{-5}\overset{2}{k}'+3r^{-6}\overset{2}{k}) \\ r^{-1}\overset{(22)}{E}_{12}=-\tfrac{3}{2}sc(r^{-5}\overset{2}{k}'+2r^{-6}\overset{2}{k}) \\ r^{-1}\overset{(22)}{E}_{24}=\tfrac{1}{2}sc(r^{-3}\overset{2}{k}'''+3r^{-4}\overset{2}{k}''+3r^{-5}\overset{2}{k}') \end{gathered} \tag{6.4}$$

where the non-zero $(11)\times(11)$ contributions to $\overset{(22)}{E}_{ik}$ are given by

$$\begin{gathered} \overset{(22)}{E}_{11}=r^{-6}s^2\overset{1}{k}^2 \\ r^{-2}\overset{(22)}{E}_{22}=r^{-4}(-s^2\overset{1}{k}\overset{1}{k}''+2c^2\overset{1}{k}'^2)+(2-\tfrac{5}{2}s^2)(2r^{-5}\overset{1}{k}\overset{1}{k}'+r^{-6}\overset{1}{k}^2) \\ r^{-2}s^{-2}\overset{(22)}{E}_{33}=r^{-4}(s^2\overset{1}{k}\overset{1}{k}''+2c^2\overset{1}{k}'^2)+(2-\tfrac{3}{2}s^2)(2r^{-5}\overset{1}{k}\overset{1}{k}'+r^{-6}\overset{1}{k}^2) \\ \overset{(22)}{E}_{44}=r^{-2}s^2\overset{1}{k}''^2+2r^{-3}s^2\overset{1}{k}'\overset{1}{k}''+r^{-4}[s^2\overset{1}{k}\overset{1}{k}''+(2-s^2)\overset{1}{k}'^2]+(2-\tfrac{3}{2}s^2)(2r^{-5}\overset{1}{k}\overset{1}{k}'+r^{-6}\overset{1}{k}^2) \\ r^{-1}\overset{(22)}{E}_{12}=-2sc(r^{-5}\overset{1}{k}\overset{1}{k}'+r^{-6}\overset{1}{k}^2) \\ \overset{(22)}{E}_{14}=2r^{-4}c^2\overset{1}{k}'^2+4r^{-5}c^2\overset{1}{k}\overset{1}{k}'+r^{-6}(2-\tfrac{3}{2}s^2)\overset{1}{k}^2 \\ r^{-1}\overset{(22)}{E}_{24}=2sc[r^{-3}\overset{1}{k}'\overset{1}{k}''+r^{-4}(\overset{1}{k}\overset{1}{k}')'+r^{-5}\overset{1}{k}\overset{1}{k}'] \end{gathered} \tag{6.5}$$

† The $(1q)\times(1r)$ contribution to E_{ik} is to mean the part of E_{ik} emanating from the combination $\overset{(1q)}{F}_{ik}\times\overset{(1r)}{F}_{ik}$ in the second of equations (1.1), $\overset{(ps)}{F}_{ik}$ denoting the coefficient of e^pa^s in F_{ik}.

where the non-zero $(10)\times(13)$ contributions to $\overset{(23)}{E}_{ik}$ are given by

$$\begin{aligned}
r^{-2}\overset{(23)}{E}_{22} &= r^{-2}s^{-2}\overset{(23)}{E}_{33} = \overset{(23)}{E}_{44} = \overset{(23)}{E}_{14} \\
&= r^{-4}(-\tfrac{1}{3}c+\tfrac{2}{3}c^3)\overset{3}{k}{}''' + r^{-5}(-\tfrac{7}{3}c+4c^3)\overset{3}{k}{}'' + (-6c+10c^3)(r^{-6}\overset{3}{k}{}' + r^{-7}\overset{3}{k}) \\
r^{-1}\overset{(23)}{E}_{12} &= r^{-5}(-\tfrac{5}{6}s+s^3)\overset{3}{k}{}'' + (-2s+\tfrac{5}{2}s^3)(2r^{-6}\overset{3}{k}{}' + 3r^{-7}\overset{3}{k}) \\
r^{-1}\overset{(23)}{E}_{24} &= \tfrac{1}{6}r^{-3}sc^2\overset{3}{k}{}^{iv} + r^{-4}(\tfrac{5}{6}s-s^3)\overset{3}{k}{}''' + (2s-\tfrac{5}{2}s^3)(r^{-5}\overset{3}{k}{}'' + r^{-6}\overset{3}{k}{}')
\end{aligned} \tag{6.6}$$

and, finally, where the non-zero $(11)\times(12)$ contributions to $\overset{(23)}{E}_{ik}$ are given by

$$\begin{aligned}
\overset{(23)}{E}_{11} &= 3s^2c(r^{-6}\overset{1}{k}\overset{2}{k}{}' + 2r^{-7}\overset{1}{k}\overset{2}{k}) \\
r^{-2}\overset{(23)}{E}_{22} &= r^{-4}[-\tfrac{3}{2}s^2c\overset{1}{k}{}''\overset{2}{k}{}' + (-c+3c^3)\overset{1}{k}{}'\overset{2}{k}{}'' - \tfrac{1}{2}s^2c\overset{1}{k}\overset{2}{k}{}'''] \\
&\quad + r^{-5}[-3s^2c\overset{1}{k}{}''\overset{2}{k} + (-\tfrac{9}{2}c+\tfrac{21}{2}c^3)\overset{1}{k}{}'\overset{2}{k}{}' + (-\tfrac{5}{2}c+\tfrac{9}{2}c^3)\overset{1}{k}\overset{2}{k}{}''] \\
&\quad + (-6c+12c^3)[r^{-6}(\overset{1}{k}\overset{2}{k})' + r^{-7}\overset{1}{k}\overset{2}{k}] \\
r^{-2}s^{-2}\overset{(23)}{E}_{33} &= r^{-4}[\tfrac{3}{2}s^2c\overset{1}{k}{}''\overset{2}{k}{}' + (-c+3c^3)\overset{1}{k}{}'\overset{2}{k}{}'' + \tfrac{1}{2}s^2c\overset{1}{k}\overset{2}{k}{}'''] \\
&\quad + r^{-5}[3s^2c\overset{1}{k}{}''\overset{2}{k} + (-\tfrac{3}{2}c+\tfrac{15}{2}c^3)\overset{1}{k}{}'\overset{2}{k}{}' + (\tfrac{1}{2}c+\tfrac{3}{2}c^3)\overset{1}{k}\overset{2}{k}{}''] \\
&\quad + 6c^3[r^{-6}(\overset{1}{k}\overset{2}{k})' + r^{-7}\overset{1}{k}\overset{2}{k}] \\
\overset{(23)}{E}_{44} &= r^{-2}s^2c\overset{1}{k}{}''\overset{2}{k}{}''' + r^{-3}s^2c(3\overset{1}{k}{}''\overset{2}{k}{}'' + \overset{1}{k}{}'\overset{2}{k}{}''') + r^{-4}(\tfrac{9}{2}s^2c\overset{1}{k}{}''\overset{2}{k}{}' + 2c\overset{1}{k}{}'\overset{2}{k}{}'' + \tfrac{1}{2}s^2c\overset{1}{k}\overset{2}{k}{}''') \\
&\quad + r^{-5}[3s^2c\overset{1}{k}{}''\overset{2}{k} + (\tfrac{3}{2}c+\tfrac{9}{2}c^3)\overset{1}{k}{}'\overset{2}{k}{}' + (\tfrac{1}{2}c+\tfrac{3}{2}c^3)\overset{1}{k}\overset{2}{k}{}''] + 6c^3[r^{-6}(\overset{1}{k}\overset{2}{k})' + r^{-7}\overset{1}{k}\overset{2}{k}] \\
r^{-1}\overset{(23)}{E}_{12} &= r^{-5}[-3sc^2\overset{1}{k}{}'\overset{2}{k}{}' + (-s+\tfrac{3}{2}s^3)\overset{1}{k}\overset{2}{k}{}''] \\
&\quad + r^{-6}[-6sc^2\overset{1}{k}{}'\overset{2}{k} + (-6s+\tfrac{15}{2}s^3)\overset{1}{k}\overset{2}{k}{}'] + r^{-7}(-9s+\tfrac{21}{2}s^3)\overset{1}{k}\overset{2}{k} \\
\overset{(23)}{E}_{14} &= r^{-4}(-c+3c^3)\overset{1}{k}{}'\overset{2}{k}{}'' + r^{-5}(-c+3c^3)(3\overset{1}{k}{}'\overset{2}{k}{}' + \overset{1}{k}\overset{2}{k}{}'') \\
&\quad + r^{-6}[(-3c+9c^3)\overset{1}{k}{}'\overset{2}{k} + (-\tfrac{3}{2}c+\tfrac{15}{2}c^3)\overset{1}{k}\overset{2}{k}{}'] + 6r^{-7}c^3\overset{1}{k}\overset{2}{k} \\
r^{-1}\overset{(23)}{E}_{24} &= r^{-3}[(s-\tfrac{3}{2}s^3)\overset{1}{k}{}''\overset{2}{k}{}'' + sc^2\overset{1}{k}{}'\overset{2}{k}{}'''] + r^{-4}[(3s-\tfrac{9}{2}s^3)\overset{1}{k}{}''\overset{2}{k}{}' + (4s-\tfrac{9}{2}s^3)\overset{1}{k}{}'\overset{2}{k}{}'' + sc^2\overset{1}{k}\overset{2}{k}{}'''] \\
&\quad + r^{-5}[(3s-\tfrac{9}{2}s^3)\overset{1}{k}{}''\overset{2}{k} + (6s-\tfrac{15}{2}s^3)\overset{1}{k}{}'\overset{2}{k}{}' + 3sc^2\overset{1}{k}\overset{2}{k}{}''] \\
&\quad + r^{-6}[(3s-\tfrac{9}{2}s^3)\overset{1}{k}{}'\overset{2}{k} + 3sc^2\overset{1}{k}\overset{2}{k}{}'].
\end{aligned} \tag{6.7}$$

From the above formulae we can obtain the Galilean components K^{α}_{G} of the outward flow of linear momentum of radiation from the source in § 2. (The subscript G refers to Galilean coordinates.) It is fairly obvious from the symmetry of the linear source that there is no flux of the x or y component of the momentum, and this can be verified by direct calculation (at least as far as the (23) approximation). So attention will be confined to the z component K_G^3.

The rate at which linear momentum flows out of a large sphere S, centre the origin and radius r, is given (in the linear approximation) by

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_G^3) = \lim_{r\to\infty} \int_S E_G^{3\alpha} n_\alpha \, \mathrm{d}S. \tag{6.8}$$

Transforming from Galilean coordinates (x, y, z, t) to the coordinates (r, θ, ϕ, u) of the Bondi metric and using equation (3.6) we readily find for the rate at which linear momentum flows out of an infinite sphere, centre O, the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(K_G^3) = \lim_{r\to\infty} r^2 \int (E^{11}\cos\theta - E^{12} r \sin\theta) \, \mathrm{d}\Omega \tag{6.9}$$

where

$$\int f \, \mathrm{d}\Omega \overset{\text{def}}{=} \int_0^{2\pi} \int_0^{\pi} f \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{6.10}$$

Employing equations (4.3), (5.6), (6.1)–(6.7) in equation (6.9) yields

$$[K_G^3]_{u=-\infty}^{\infty} = e^2\left(\frac{8}{15}\pi a^3 \int_{u_1}^{u_2} \overset{1}{k}{}'' \overset{2}{k}{}''' \, \mathrm{d}u + \mathrm{O}(a^4)\right) + \mathrm{O}(e^3) \tag{6.11}$$

as the total outward flow of momentum of radiation from the linear source. This being of order e^2a^3 leads us to expect terms to appear in the (23) approximation which reveal that the linear source itself undergoes a generally permanent variation in momentum in the z direction equal and opposite to the momentum specified by the leading, e^2a^3, part of the expression on the right of equation (6.11). This will be confirmed in the next section.

7. The second approximations. Change of linear momentum of the source in the (23) approximation

Every $(2s)$ approximation is the set of equations (A.6)–(A.12) of appendix 2 with the quantities $\overset{(2s)}{P}, \overset{(2s)}{Q}, \ldots, \overset{(2s)}{N}$ on the right given by

$$\begin{aligned}
&\overset{(2s)}{P} = \alpha \overset{(2s)}{E}_{11} && \overset{(2s)}{Q} = \alpha r^{-2} \overset{(2s)}{E}_{22} && \\
&\overset{(2s)}{R} = \alpha r^{-2} s^{-2} \overset{(2s)}{E}_{33} && \overset{(2s)}{S} = \alpha \overset{(2s)}{E}_{44} && \\
&\overset{(2s)}{L} = \alpha r^{-1} \overset{(2s)}{E}_{12} && \overset{(2s)}{M} = \alpha \overset{(2s)}{E}_{14} && \overset{(2s)}{N} = \alpha r^{-1} \overset{(2s)}{E}_{24}
\end{aligned} \tag{7.1}$$

in which

$$\alpha \overset{\text{def}}{=} -16\pi. \tag{7.2}$$

The corresponding $(2s)$ solution is determined by equations (A.13)–(A.16).

On insertion of the formulae (6.2)–(6.7), one set at a time, in equations (7.1) and on use of equations (A.13)–(A.16), the following approximate solutions can eventually be found for the linear source in § 2:

7.1. The (20) *(Nordström) solution*

$$\overset{(20)}{D} = 4\pi r^{-2}. \tag{7.3}$$

7.2. The (21) *solution*

$$\overset{(21)}{D} = \alpha c(-\tfrac{2}{3}r^{-2}\overset{1}{k}' - \tfrac{1}{2}r^{-3}\overset{1}{k}) \qquad \overset{(21)}{G} = \alpha s(-\tfrac{1}{3}r^{-2}\overset{1}{k}' + \tfrac{1}{4}r^{-3}\overset{1}{k}). \tag{7.4}$$

7.3. The (22) *solution corresponding to the* $(10)\times(12)$ *contribution to* E_{ik}

$$\begin{aligned}
\overset{(22)}{B} &= -\overset{(22)}{C} = \alpha s^2(-\tfrac{1}{24}r^{-3}\overset{2}{k}' + \tfrac{1}{8}r^{-4}\overset{2}{k}) \\
\overset{(22)}{D} &= \alpha(2-3s^2)(-\tfrac{1}{6}r^{-2}\overset{2}{k}'' - \tfrac{1}{3}r^{-3}\overset{2}{k}' - \tfrac{1}{4}r^{-4}\overset{2}{k}) \\
\overset{(22)}{G} &= \alpha sc(-\tfrac{1}{6}r^{-2}\overset{2}{k}'' + \tfrac{1}{4}r^{-3}\overset{2}{k}' + \tfrac{1}{2}r^{-4}\overset{2}{k}).
\end{aligned} \tag{7.5}$$

7.4. The (22) *solution corresponding to the* $(11)\times(11)$ *contribution to* E_{ik}

$$\begin{aligned}
\overset{(22)}{B} &= -\overset{(22)}{C} = \alpha s^2\left[r^{-1}\left(\frac{1}{12}\int_{-\infty}^{u} \overset{1}{k}''^2\,\mathrm{d}u - \tfrac{1}{3}\overset{1}{k}'\overset{1}{k}''\right) - \tfrac{1}{4}r^{-3}\overset{1}{k}\overset{1}{k}'\right] \\
\overset{(22)}{D} &= \alpha\Bigg[r^{-1}\left(-\frac{1}{3}\int_{-\infty}^{u} \overset{1}{k}''^2\,\mathrm{d}u + (\tfrac{4}{3} - 2s^2)\overset{1}{k}'\overset{1}{k}''\right) \\
&\qquad -\tfrac{1}{2}r^{-2}s^2\overset{1}{k}'^2 + r^{-3}(-\tfrac{1}{2} + \tfrac{1}{4}s^2)\overset{1}{k}\overset{1}{k}' - \tfrac{1}{4}r^{-4}c^2\overset{1}{k}^2\Bigg] \\
\overset{(22)}{F} &= \tfrac{1}{16}\alpha r^{-4}s^2\overset{1}{k}^2 \\
\overset{(22)}{G} &= \alpha sc\left[r^{-1}\left(-\frac{1}{6}\int_{-\infty}^{u} \overset{1}{k}''^2\,\mathrm{d}u + \tfrac{2}{3}\overset{1}{k}'\overset{1}{k}''\right) - r^{-2}\overset{1}{k}'^2 - \tfrac{1}{4}r^{-3}\overset{1}{k}\overset{1}{k}' + \tfrac{1}{8}r^{-4}\overset{1}{k}^2\right].
\end{aligned} \tag{7.6}$$

7.5. The (23) *solution corresponding to the* $(10)\times(13)$ *contribution to* E_{ik}

$$\begin{aligned}
\overset{(23)}{B} &= -\overset{(23)}{C} = \alpha s^2 c(-\tfrac{1}{36}r^{-3}\overset{3}{k}'' + \tfrac{1}{8}r^{-4}\overset{3}{k}' + \tfrac{3}{8}r^{-5}\overset{3}{k}) \\
\overset{(23)}{D} &= \alpha[r^{-2}(\tfrac{1}{9}c - \tfrac{2}{9}c^3)\overset{3}{k}''' + r^{-3}(\tfrac{1}{2}c - \tfrac{31}{36}c^3)\overset{3}{k}'' + (3c - 5c^3)(\tfrac{19}{60}r^{-4}\overset{3}{k}' + \tfrac{1}{4}r^{-5}\overset{3}{k})] \\
\overset{(23)}{G} &= \alpha[-\tfrac{1}{18}r^{-2}sc^2\overset{3}{k}''' + r^{-3}(\tfrac{1}{8}s - \tfrac{7}{48}s^3)\overset{3}{k}'' + (4s - 5s^3)(\tfrac{3}{20}r^{-4}\overset{3}{k}' + \tfrac{3}{16}r^{-5}\overset{3}{k})].
\end{aligned} \tag{7.7}$$

7.6. The (23) *solution corresponding to the* (11)×(12) *contribution to* E_{ik}

$$
\begin{aligned}
\overset{(23)}{B} = -\overset{(23)}{C} = \alpha s^2 c[& r^{-1}(\tfrac{1}{20}Y - \tfrac{1}{20}\overset{1}{k}{}'''\overset{2}{k}{}' - \tfrac{3}{20}\overset{1}{k}{}''\overset{2}{k}{}''' - \tfrac{7}{60}\overset{1}{k}{}'\overset{2}{k}{}''') \\
& + r^{-3}(-\tfrac{5}{8}\overset{1}{k}{}'\overset{2}{k}{}' - \tfrac{1}{8}\overset{1}{k}\overset{2}{k}{}'') + r^{-4}(-\tfrac{1}{2}\overset{1}{k}{}'\overset{2}{k} - \tfrac{3}{16}\overset{1}{k}\overset{2}{k}{}') + \tfrac{3}{40}r^{-5}\overset{1}{k}\overset{2}{k}]
\end{aligned}
$$

$$
\begin{aligned}
\overset{(23)}{D} = \alpha\{ & r^{-1}[-\tfrac{1}{5}cY + (-\tfrac{3}{10}c + \tfrac{1}{2}c^3)\overset{1}{k}{}'''\overset{2}{k}{}' + (-\tfrac{9}{10}c + \tfrac{3}{2}c^3)\overset{1}{k}{}''\overset{2}{k}{}'' + (-\tfrac{7}{10}c + \tfrac{7}{6}c^3)\overset{1}{k}{}'\overset{2}{k}{}'''] \\
& + r^{-2}[-\tfrac{1}{15}cZ - \tfrac{2}{15}cE + (-\tfrac{3}{5}c + c^3)\overset{1}{k}{}''\overset{2}{k}{}' + (-\tfrac{37}{30}c + \tfrac{3}{2}c^3)\overset{1}{k}{}'\overset{2}{k}{}''] \\
& + r^{-3}[(-\tfrac{9}{8}c + \tfrac{7}{8}c^3)\overset{1}{k}{}'\overset{2}{k}{}' + (-\tfrac{1}{8}c - \tfrac{1}{8}c^3)\overset{1}{k}\overset{2}{k}{}''] \\
& + r^{-4}[(-\tfrac{3}{10}c - \tfrac{1}{10}c^3)\overset{1}{k}{}'\overset{2}{k} - \tfrac{3}{5}c^3\overset{1}{k}\overset{2}{k}{}'] + r^{-5}(\tfrac{1}{4}c - \tfrac{3}{4}c^3)\overset{1}{k}\overset{2}{k}\}
\end{aligned}
\tag{7.8}
$$

$$
\overset{(23)}{F} = \alpha s^2 c(\tfrac{3}{16}r^{-4}\overset{1}{k}\overset{2}{k}{}' + \tfrac{3}{10}r^{-5}\overset{1}{k}\overset{2}{k})
$$

$$
\begin{aligned}
\overset{(23)}{G} = \alpha\{ & r^{-1}[(-\tfrac{1}{10}s + \tfrac{1}{8}s^3)Y + (\tfrac{1}{10}s - \tfrac{1}{8}s^3)\overset{1}{k}{}'''\overset{2}{k}{}' + (\tfrac{3}{10}s - \tfrac{3}{8}s^3)\overset{1}{k}{}''\overset{2}{k}{}'' + (\tfrac{7}{30}s - \tfrac{7}{24}s^3)\overset{1}{k}{}'\overset{2}{k}{}'''] \\
& + r^{-2}[\tfrac{1}{15}sZ + \tfrac{2}{15}sE + (-\tfrac{2}{5}s + \tfrac{1}{2}s^3)\overset{1}{k}{}''\overset{2}{k}{}' + (-\tfrac{19}{15}s + \tfrac{3}{2}s^3)\overset{1}{k}{}'\overset{2}{k}{}''] \\
& + r^{-3}[(-\tfrac{9}{8}s + \tfrac{51}{32}s^3)\overset{1}{k}{}'\overset{2}{k}{}' + (-\tfrac{1}{8}s + \tfrac{3}{32}s^3)\overset{1}{k}\overset{2}{k}{}''] \\
& + r^{-4}[(-\tfrac{1}{5}s + \tfrac{2}{5}s^3)\overset{1}{k}{}'\overset{2}{k} + (\tfrac{3}{40}s - \tfrac{3}{80}s^3)\overset{1}{k}\overset{2}{k}{}'] + r^{-5}(\tfrac{7}{20}s - \tfrac{27}{80}s^3)\overset{1}{k}\overset{2}{k}\}
\end{aligned}
$$

with

$$
Y \overset{\text{def}}{=} \int_{-\infty}^{u} \overset{1}{k}{}''\overset{2}{k}{}''' \,\mathrm{d}u \qquad Z \overset{\text{def}}{=} \int_{-\infty}^{u} Y(u)\,\mathrm{d}u \qquad E \overset{\text{def}}{=} \int_{-\infty}^{u} \overset{1}{k}{}''\overset{2}{k}{}''\,\mathrm{d}u. \tag{7.9}
$$

The solutions (7.3)–(7.5) and (7.7) were found without the use of the five functions (A.17) of integration, and these functions were put equal to zero. However for obtaining the solutions (7.6) and (7.8) the respective values

$$
\chi = \alpha\left((-\tfrac{2}{3} + \tfrac{1}{2}s^2)\int_{-\infty}^{u} \overset{1}{k}{}''^2\,\mathrm{d}u + (\tfrac{8}{3} - 4s^2)\overset{1}{k}{}'\overset{1}{k}{}''\right) \tag{7.10}
$$

$$
\chi = \alpha[(\tfrac{1}{10}c - \tfrac{1}{2}c^3)Y + (3c - 5c^3)(-\tfrac{1}{5}\overset{1}{k}{}'''\overset{2}{k}{}' - \tfrac{3}{5}\overset{1}{k}{}''\overset{2}{k}{}'' - \tfrac{7}{15}\overset{1}{k}{}'\overset{2}{k}{}''')] \tag{7.11}
$$

had to be assigned to the second function χ of integration, the other four being ignored. This was done to ensure that the solutions (7.6) and (7.8) satisfied the regularity conditions for all θ, u and for all $r > 0$, except for possible gravitational dipole terms linear in u representing a recoil of the source off to infinity.

To ascertain such a recoil, we shall examine in the leading (2s) solutions (7.3)–(7.8) non-transient terms of orders r^{-1} and r^{-2}, i.e. terms of these orders that generally undergo permanent changes with u†. Only the (22) and (23) solutions (7.6) and (7.8)

† Following the course adopted by Bonnor and Rotenberg (1966) in the purely gravitational case, we proceed as far as terms of order r^{-2} because the gravitational dipole terms representing a recoil involve r^{-2}: see appendix 3.

contain terms of this sort involving integrals. Such terms in the (22) solution (7.6) represent a change in mass of the source, as shown by Rotenberg (1966), and this does not concern us. Picking out the non-transient terms of orders r^{-1} and r^{-2} from the (23) solution (7.8), we obtain

$$
\begin{aligned}
&\overset{(23)}{B} = -\overset{(23)}{C} = \tfrac{1}{20}\alpha r^{-1}s^2 cY \\
&\overset{(23)}{D} = \alpha[-\tfrac{1}{5}r^{-1}cY - \tfrac{1}{15}r^{-2}c(Z+2E)] \\
&\overset{(23)}{G} = \alpha[r^{-1}(-\tfrac{1}{10}s + \tfrac{1}{8}s^3)Y + \tfrac{1}{15}r^{-2}s(Z+2E)].
\end{aligned}
\tag{7.12}
$$

Like all other terms in r^{-1} and r^{-2} of the leading ($2s$) solutions (7.3)–(7.8), the above terms vanish for $u < u_1$ (beginning of the motion of the source). However, for $u > u_2$ (end of the motion) the integral Y, defined by the first of equations (7.9), is a constant, and we write

$$
Y = Y_0 \overset{\text{def}}{=} \int_{u_1}^{u_2} \overset{1}{k}''\overset{2}{k}'''\,\mathrm{d}u \qquad (u > u_2). \tag{7.13}
$$

Consequently, from the second and third of the definitions (7.9), we have for $u > u_2$

$$
\begin{aligned}
Z+2E &= \int_{u_1}^{u} [Y(w) + 2\overset{1}{k}''(w)\overset{2}{k}''(w)]\,\mathrm{d}w \\
&= \int_{u_1}^{u_2} [Y(w) + 2\overset{1}{k}''(w)\overset{2}{k}''(w)]\,\mathrm{d}w + \int_{u_2}^{u} Y_0\,\mathrm{d}w \\
&= \int_{u_1}^{u_2} [Y(u) + 2\overset{1}{k}''(u)\overset{2}{k}''(u)]\,\mathrm{d}u + Y_0(u - u_2).
\end{aligned}
$$

So we get

$$
Z + 2E = Y_0(u + u_0) \qquad (u > u_2) \tag{7.14}
$$

where u_0 is a constant given by

$$
u_0 Y_0 = -u_2 Y_0 + \int_{u_1}^{u_2} (Y + 2\overset{1}{k}''\overset{2}{k}'')\,\mathrm{d}u. \tag{7.15}
$$

Inserting equations (7.13) and (7.14) in equations (7.12) we find that, for $u > u_2$,

$$
\begin{aligned}
&\overset{(23)}{B} = -\overset{(23)}{C} = \tfrac{1}{20}\alpha Y_0 r^{-1}s^2 c \\
&\overset{(23)}{D} = \alpha Y_0[-\tfrac{1}{5}r^{-1}c - \tfrac{1}{15}r^{-2}c(u+u_0)] \\
&\overset{(23)}{G} = \alpha Y_0[r^{-1}(-\tfrac{1}{10}s + \tfrac{1}{8}s^3) + \tfrac{1}{15}r^{-2}s(u+u_0)].
\end{aligned}
\tag{7.16}
$$

The coordinate transformation

$$
\begin{aligned}
&r = r^* + \tfrac{1}{20}e^2a^3\alpha Y_0 s^{*2}c^* \qquad \theta = \theta^* - \tfrac{1}{40}e^2a^3\alpha Y_0 r^{*-1}s^{*3} \\
&\phi = \phi^* \qquad u = u^* + \tfrac{1}{120}e^2a^3\alpha Y_0(-3c^* + c^{*3})
\end{aligned}
\tag{7.17}
$$

with $s^* \stackrel{\text{def}}{=} \sin\theta^*$, $c^* \stackrel{\text{def}}{=} \cos\theta^*$, simplifies the solution (7.16) considerably to

$$\overset{(23)}{D} = \alpha Y_0 c[-\tfrac{1}{5}r^{-1} - \tfrac{1}{15}r^{-2}(u+u_0)] \qquad \overset{(23)}{G} = \tfrac{1}{15}\alpha Y_0 r^{-2} s(u+u_0) \tag{7.18}$$

the asterisks being omitted; the transformation (7.17) does not affect the lower ($2s$) solutions.

This solution (7.18) (valid for $u > u_2$, end of the motion of the source) contains the gravitational dipole terms $-(\frac{1}{15}\alpha Y_0)r^{-2}cu$ and $(\frac{1}{15}\alpha Y_0)r^{-2}su$ in $\overset{(23)}{D}$ and $\overset{(23)}{G}$, respectively. We now readily show that this solution does in fact represent a recoil of the source off to infinity. Let us combine the approximate metric corresponding to the solution (7.18) with the Schwarzschild metric given by equation (5.3) with $e=0$, namely

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + (1-2mr^{-1})\,\mathrm{d}u^2 + 2\,\mathrm{d}r\,\mathrm{d}u \tag{7.19}$$

where m is the mass of the source. Then we obtain

$$\begin{aligned}\mathrm{d}s^2 = {} & -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \\ & + \{1 - 2mr^{-1} + e^2a^3\alpha Y_0 c[-\tfrac{1}{5}r^{-1} - \tfrac{1}{15}r^{-2}(u+u_0)]\}\,\mathrm{d}u^2 \\ & + 2\,\mathrm{d}r\,\mathrm{d}u + \tfrac{1}{15}e^2a^3\alpha Y_0 r^{-2}s(u+u_0)(2r\,\mathrm{d}\theta\,\mathrm{d}u).\end{aligned} \tag{7.20}$$

This becomes identical to the Schwarzschild linear-momentum metric (A.21) of appendix 3 when the linear momentum K is assigned the value

$$\tfrac{1}{30}e^2a^3\alpha Y_0 = -\tfrac{8}{15}\pi e^2 a^3 \int_{u_1}^{u_2} \overset{1}{k}''\overset{2}{k}'''\,\mathrm{d}u \tag{7.21}$$

by virtue of definitions (7.2) and (7.13). Hence the source will finally (after $u = u_2$) move with uniform linear momentum (7.21). This momentum of recoil, which does not generally vanish, clearly accounts for the e^2a^3 contribution of the total outward flow (6.11) of linear momentum of electromagnetic radiation.

Appendix 1. The linearized outgoing wave solution of the electromagnetic potential

Here we derive the external multipole wave solution (3.7) of the linear approximation to equations (3.1).

In Galilean coordinates (x_α, t), the potential ϕ_i for weak outgoing wave fields may be written in the Kirchhoff form

$$\phi_i = \int_V r^{*-1} J_i(\tilde{x}_\alpha, t - r^*)\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\,\mathrm{d}\tilde{x}_3 \qquad \eta^{ab}J_{a,b} = 0 \tag{A.1}$$

(Eddington 1924, § 74, Rotenberg 1966) where the integral covers any space volume V including the source of the field, and r^* is the distance of the point $\tilde{\mathrm{P}}(\tilde{x}_\alpha)$ (to which the space element $\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\,\mathrm{d}\tilde{x}_3$ of integration corresponds) from the field point $\mathrm{P}(x_\alpha)$ under consideration. Expanding the integrand in the first of equations (A.1) about $(\tilde{x}_\alpha, t-r)$ via the Taylor theorem, we have

$$\frac{1}{r^*}J_i(\tilde{x}_\alpha, t-r^*) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{g^n}{r^*}\right) J_i^{(n)}(\tilde{x}_\alpha, t-r) \tag{A.2}$$

in which

$$g \stackrel{\text{def}}{=} r^* - r \tag{A.3}$$

and the superscript (n) denotes $\partial^n/\partial t^n$. Using the binomial theorem for the expansion of g^n/r^* $(n = 0, 1, 2, \ldots)$ in series of ascending powers of $\tilde{r}/r$ valid in the range $r > \tilde{r} = \mathrm{O}\tilde{\mathrm{P}} = (\tilde{x}_\sigma \tilde{x}_\sigma)^{1/2}$, we obtain

$$\frac{1}{r^*} = \frac{1}{r} \sum_{n=0}^{\infty} \frac{\tilde{r}^n}{r^n} \mathrm{P}_n(\cos\theta^*) \qquad \frac{g}{r^*} = -\sum_{n=1}^{\infty} \frac{\tilde{r}^n}{r^n} \mathrm{P}_n(\cos\theta^*)$$

$$\frac{g^2}{r^*} = r\left\{\frac{\tilde{r}^2}{r^2}\cos^2\theta^* + \frac{\tilde{r}^3}{r^3}(2\cos^3\theta^* - \cos\theta^*) + \mathrm{O}\left[\left(\frac{\tilde{r}}{r}\right)^4\right]\right\}$$

$$\frac{g^3}{r^*} = r^2\left\{-\frac{\tilde{r}^3}{r^3}\cos^3\theta^* + \mathrm{O}\left[\left(\frac{\tilde{r}}{r}\right)^4\right]\right\} \tag{A.4}$$

$$\vdots$$

$$\frac{g^n}{r^*} = r^{n-1}\mathrm{O}\left[\left(\frac{\tilde{r}}{r}\right)^n\right]$$

$$\vdots$$

where θ^* is the angle $\mathrm{PO}\tilde{\mathrm{P}}$ and P_n are the Legendre polynomials. Inserting the expansions (A.4) in the expansion (A.2), employing the formulae

$$\tilde{r}^2 = \tilde{x}_\sigma \tilde{x}_\sigma \qquad \cos\theta^* = n_\sigma \tilde{x}_\sigma/\tilde{r} \qquad (n_\sigma = x_\sigma/r) \tag{A.5}$$

substituting the result in the first of equations (A.1) and using equation (3.2) and the notations (3.3) and (3.5) finally lead to the multipole wave solution (3.7) for ϕ_i.

Appendix 2. The approximate Einstein–Maxwell equations for the Bondi metric, and their solution

Using the expansions (5.4) and (4.2) in the first of equations (1.1) we obtain the (ps) approximation, in the coordinates of the Bondi metric (5.1), as the seven equations below (in which $R'_{ik} \stackrel{\text{def}}{=} R_{ik} + 8\pi E_{ik}$). To save printing, the symbol (ps), which should have been placed above each capital letter in these equations, has been omitted throughout this appendix, except where confusion may result without it.

$$2R'_{11} = 0: \qquad -4r^{-1}F_1 = P \tag{A.6}$$

$$\begin{aligned} 2r^{-2}R'_{22} = 0: \quad & B_{11} - 2B_{14} + 2r^{-1}(B_1 - B_4 + D_1 - F_1 - G_{12}) \\ & + r^{-2}(-B_{22} - 3B_2\cot\theta + 2B + 2D + 2F_{22} - 4F - 4G_2 - 2G\cot\theta) = Q \end{aligned} \tag{A.7}$$

$$\begin{aligned} 2r^{-2}s^{-2}R'_{33} = 0: \quad & -B_{11} + 2B_{14} + 2r^{-1}(-B_1 + B_4 + D_1 - F_1 - G_1\cot\theta) \\ & + r^{-2}(-B_{22} - 3B_2\cot\theta + 2B + 2D + 2F_2\cot\theta - 4F \\ & -2G_2 - 4G\cot\theta) = R \end{aligned} \tag{A.8}$$

$$\begin{aligned} 2R'_{44} = 0: \quad & -D_{11} + 2F_{14} + 2r^{-1}(-D_1 - D_4 + 2F_4 + G_{24} + G_4\cot\theta) \\ & - r^{-2}(D_{22} + D_2\cot\theta) = S \end{aligned} \tag{A.9}$$

$$2r^{-1}R'_{12}=0: \qquad -G_{11}+r^{-1}(-B_{12}-2B_1\cot\theta+F_{12}-2G_1)+2r^{-2}(-F_2+G)=L \tag{A.10}$$

$$2R'_{14}=0: \qquad -D_{11}+2F_{14}+r^{-1}(-2D_1+G_{12}+G_1\cot\theta) +r^{-2}(-F_{22}-F_2\cot\theta+G_2+G\cot\theta)=M \tag{A.11}$$

$$2r^{-1}R'_{24}=0: \qquad -G_{11}+G_{14}+r^{-1}(-B_{24}-2B_4\cot\theta-D_{12}+F_{12}+F_{24}-2G_1-G_4)=N. \tag{A.12}$$

In the above equations a subscript 1, 2 or 4 after B, D, F or G means differentiation with respect to r, θ or u, respectively; this is to apply to any non-tensorial symbol, unless the context implies otherwise. The second of equations (5.1) has been employed, so that C is absent from the above equations. The terms linear in $\overset{(ps)}{g}_{ik}$ and their derivatives appear explicitly on the left of these equations, whereas the terms non-linear in $\overset{(qr)}{g}_{ik}$ and their derivatives, determined from previous approximations, accompany $\overset{(ps)}{E}_{ik}$ to form the quantities $P, \ldots, N$ on the right.

The formal solution of equations (A.6)–(A.12), which has been derived in papers by Bonnor, Hunter and Rotenberg (Bonnor and Rotenberg 1966, Rotenberg 1966, Hunter and Rotenberg 1969), is given by

$$F=-\tfrac{1}{4}\int rP\,\mathrm{d}r+\eta(\theta,u) \tag{A.13}$$

$$\begin{aligned}\Box' D &\overset{\text{def}}{=} D_{11}-2D_{14}+2r^{-1}(D_1+D_4)+r^{-2}(D_{22}+D_2\cot\theta)\\ &=-S+2(F_{14}+2r^{-1}F_4)\\ &\quad+2r^{-2}\left(\int [r^2(M-2F_{14})+(F_{22}+F_2\cot\theta)]\,\mathrm{d}r+\chi(\theta,u)\right)_4\end{aligned} \tag{A.14}$$

$$G=r^{-1}\int F_2\,\mathrm{d}r+r^{-1}\operatorname{cosec}\theta\int \sin\theta\left(\int r^2(M-2F_{14})\,\mathrm{d}r+r^2D_1+\chi\right)\mathrm{d}\theta+\nu(r,u)\operatorname{cosec}\theta \tag{A.15}$$

$$\begin{aligned}B&=\operatorname{cosec}^2\theta\int \sin^2\theta\left(-\int [rL+2r^{-1}(F_2-G)]\,\mathrm{d}r+F_2-G-rG_1\right)\mathrm{d}\theta\\ &\quad+\tau(r,u)\operatorname{cosec}^2\theta+\mu(\theta,u)\end{aligned} \tag{A.16}$$

in which

$$\eta(\theta,u) \qquad \chi(\theta,u) \qquad \nu(r,u) \qquad \tau(r,u) \qquad \mu(\theta,u) \tag{A.17}$$

are five functions of integration. These arbitrary functions must be chosen to ensure that the solution satisfies *all* the seven equations (A.6)–(A.12), and that it is Galilean at spatial infinity and regular everywhere except at O and except for possible gravitational dipole terms (proportional to u), which represent a recoil of the source. Special precaution must be taken to avoid the solution from being singular in θ along the polar axis $\theta=0$. A sufficient condition for the (ps) metric to be regular in θ along the polar axis is that

$$B \operatorname{cosec}^2 \theta, C \operatorname{cosec}^2 \theta, D, F, G \operatorname{cosec} \theta \text{ be of class } C^2 \text{ near } \sin\theta = 0. \tag{A.18}$$

All the leading $(2s)$ solutions found in § 7 meet this condition.

Appendix 3. The Schwarzschild linear-momentum field and the dipole solution

Here we apply to the Schwarzschild metric

$$ds^2 = -r^{*2}(d\theta^{*2} + \sin^2\theta^* \, d\phi^{*2}) + (1 - 2mr^{*-1})\, du^{*2} + 2\, dr^*\, du^* \tag{A.19}$$

the linearized Lorentz transformation†

$$\begin{aligned} r^* &= r + vc(r + u + u_0) \qquad & \theta^* &= \theta - vsr^{-1}(r + u + u_0) \\ \phi^* &= \phi & u^* &= u - vc(u + u_0) \end{aligned} \tag{A.20}$$

where v and u_0 are constants and the relative velocity v is so small that v^n, $n \geq 2$, may be ignored. After some calculation we obtain the 'Schwarzschild linear-momentum' metric

$$\begin{aligned} ds^2 = {} & -r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \{1 - 2mr^{-1} - 2Kc[3r^{-1} + r^{-2}(u + u_0)]\}\, du^2 \\ & + 2\, dr\, du + 2Kr^{-2}s(u + u_0)(2r\, d\theta\, du) \end{aligned} \tag{A.21}$$

representing the gravitational field of a central particle of mass m moving the uniform linear momentum $K = -mv$ along the polar axis $\theta = 0$ of the (r, θ, ϕ, u) frame.

It can be verified that the contribution involving K in the metric (A.21) satisfies the linear approximation to the gravitational field equations $R_{ik} = 0$, namely equations (A.6)–(A.12) with $P, \ldots, N$ on the right equal to zero. This K contribution is referred to as the *gravitational dipole solution*. Like the monopole, Schwarzschild, solution, it does not represent gravitational waves.

A special feature of the gravitational dipole solution is that it diverges linearly with u; more previsely, it involves the 'gravitational dipole' terms $r^{-2}cu$ and $r^{-2}su$. Thus, in determining a linear momentum recoil of the electromagnetic source of § 2, we examine the metric as far as terms of order r^{-2} to seek expressions behaving like these dipole terms for $u > u_2$ (end of the motion of the source).

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† That the transformation (A.20) is a linearized Lorentz transformation, in which the (r, θ, ϕ, u) frame moves with uniform velocity v relative to the $(r^*, \theta^*, \phi^*, u^*)$ frame along the common polar axis $\theta = \theta^* = 0$, has been shown by Bonnor and Rotenberg (1966).